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On Spin(7) holonomy metric based on SU(3)/$U(1)$: I[☆]

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Abstract

We investigate the Spin(7) holonomy metric of cohomogeneity one with the principal orbit SU(3)/$U(1)$. A choice of $U(1)$ in the two-dimensional Cartan subalgebra is left as free and this allows us to manifest $\Sigma_3 = W(\mathrm{SU}(3))$ (=the Weyl group) symmetric formulation. We find asymptotically locally conical (ALC) metrics as octonionic gravitational instantons. These ALC metrics have orbifold singularities in general, but a particular choice of the $U(1)$ subgroup gives a new regular metric of Spin(7) holonomy. Complex projective space **CP**(2) that is a supersymmetric four-cycle appears as a singular orbit. A perturbative analysis of the solution near the singular orbit shows an evidence of a more general family of ALC solutions. The global topology of the manifold depends on a choice of the $U(1)$ subgroup. We also obtain an L^2-normalizable harmonic 4-form in the background of the ALC metric.

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1. Introduction

In supersymmetric compactifications of superstrings and M theory the compact manifold must allow parallel spinors and hence has a special holonomy. Among manifolds of

☆ After we submitted this paper to e-prints archives, we noticed a new paper [21], which has a considerable overlap with our paper. Many of our results have also been obtained in Section 3 of [21] and we find a complete agreement. Later we also received another preprint [22] which studies M theory on Spin(7) manifolds. In [22] the regular solution obtained as a special case of our ALC solutions is constructed independently and employed in M theory compactification.

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special holonomy, the holonomy groups G_2 in seven dimensions and Spin(7) in eight dimensions are exceptional ones. Recently, compactification of M theory on G_2 manifold has been discussed extensively in connection with $N = 1$ supersymmetric gauge theory in four dimensions [1–5]. Though it is less studied, the geometry of Spin(7) manifold is relevant to three-dimensional $N = 1$ Yang–Mills theory. Manifolds of exceptional holonomy with branes wrapping on a supersymmetric cycle are also useful for establishing the gravity/gauge theory correspondence that generalizes the AdS/CFT correspondence. In addition to few basic examples of G_2 and Spin(7) metrics on the total space of vector bundles [6,7], we now have an increasing list of explicit metrics [8–14]. All these examples are metrics on non-compact manifolds and of cohomogeneity one. There is a (rigid) supersymmetric cycle and the non-compact manifold may be identified as the normal bundle of the supersymmetric cycle. In studying the dynamics of supersymmetric compactifications of superstring and M theory, we are especially interested in the behavior when the manifold develops singularities. Potentially there are various types of singularities, but an important class of singularities in supersymmetric dynamics is the isolated conical singularity that arises when a supersymmetric cycle is shrinking. In such cases the stringy geometry is believed to be governed by a tubular neighborhood of the singularity or the shrinking supersymmetric cycle, where the above explicit metrics on the normal bundle are available. Furthermore, the geometry of such metrics often shows some universal feature that is independent of the way singularities or supersymmetric cycles are embedded in a manifold of special holonomy. The geometry of ADE singularities in $K3$ surface and the conifold transition in Calabi–Yau threefold are typical examples and we expect it is also the case with exceptional holonomy.

Let us review the basic geometry of manifolds of cohomogeneity one, following [15–17]. A Riemannian manifold (M, g) is called cohomogeneity one, if there is an isometric action on M of a Lie group G with generic orbit of codimension one. The generic orbit G/K is called principal orbit. There is an open interval I in real numbers with coordinate t, such that $\tilde{M} := I \times G/K$ is an open dense subset in M. The compliment of $\tilde{M}$ consists of singular orbits of lower dimensions, where we have a larger isotropy subgroup H ($K \subset H \subset G$). A tubular neighborhood of the singular orbit $Q = G/H$ is diffeomorphic to an open disk bundle of the normal bundle of Q. Then the principal orbits are the hypersurfaces which are the sphere bundles over Q. This means H/K is diffeomorphic to a sphere S^k. Thus, as the radius of the sphere tends to zero, the principal orbits collapse to the singular orbit. Furthermore, the existence of a smooth complete metric on the normal bundle implies that the singular orbit must be a minimal submanifold. We see the metric of cohomogeneity one is well suited for describing the geometry of collapsing supersymmetric cycles by identifying its normal bundle with a manifold of cohomogeneity one. To find out explicit metrics we begin with the fact that on $\tilde{M}$ the metric g takes the following form:

$$g = \mathrm{d}t^2 + g_t, \tag{1.1}$$

where the interval I becomes a geodesic line. For each fixed "time" t, g_t is a homogeneous metric of the principal orbit G/K. Hence if we assume that the metric is of cohomogeneity one, the condition of Ricci-flatness, or the Einstein equation in general, is reduced to a system of non-linear ordinary differential equations with respect to the transverse coordinate t to the principal orbit.

In this paper, we consider eight-dimensional metrics of cohomogeneity one with the principal orbit $\mathrm{SU}(3)/U(1)$. Part of our analysis is quite parallel to the case with the principal orbit $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ which has been worked out in [10], but there is a new feature that arises from the Weyl group symmetry $\Sigma_3 = W(\mathrm{SU}(3))$. We shall pay attention to this symmetry. In Section 2, we derive a first-order system of non-linear differential equations from the octonionic self-duality of the spin connection. If we choose vielbein (or metric ansatz) appropriately, the octonionic self-duality of the spin connection implies an existence of covariantly constant 4-form which characterizes $\mathrm{Spin}(7)$ holonomy. We also show that there is a superpotential which implies the first-order system. In Section 3, we present special solutions which give asymptotically locally conical (ALC) [10] metrics. Our ansatz for special solution was motivated by the one in [13]. Compared with ALC solutions in [10,13], our solution takes more general form to keep the Weyl group Σ_3 symmetry manifest. The singular orbit of $\mathrm{SU}(3)/U(1)$ model is the complex projective space $\mathbf{CP}(2) = \mathrm{SU}(3)/U(2)$ which is self-dual Einstein but not spin. This is in a sharp difference from the $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ case whose singular orbit is the four-dimensional sphere $S^4 = \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ which is self-dual Einstein and spin [6]. Thus the issue of global topology is more subtle in our case. We make a perturbative analysis around the singular orbit in Section 4 and find one more parameter in addition to the scale parameter in the explicit ALC solutions in Section 3. This additional parameter is an evidence for the existence of non-trivial deformation of our ALC metrics and numerical simulations support it. In Section 5, we discuss the global topology that depends on a choice of the embedding of $U(1)$ subalgebra. In general the fiber over the singular orbit $\mathbf{CP}(2)$ is the Lens space $S^3/\mathbf{Z}_n$ which leads to orbifold singularities. But there is a particular choice of $U(1)$ embedding which is free from orbifold singularities. With this choice of $U(1)$ subalgebra, our ALC solution gives a new $\mathrm{Spin}(7)$ metric on a vector bundle over $\mathbf{CP}(2)$. Finally, Section 6 is devoted to the construction of L^2-normalizable harmonic 4-forms in the background of ALC metrics, which we can employ in constructing supersymmetric M2-branes [9–11].

2. Octonionic instanton equation

We consider an eight-dimensional metric of cohomogeneity one with the principal orbit $\mathrm{SU}(3)/U(1)$. It is convenient to describe homogeneous metric in terms of the Maurer–Cartan forms of $\mathrm{SU}(3)$. The Maurer–Cartan equation is presented in Appendix A. We take a basis T_A, T_B of the Cartan part and $\sigma_{1,2}$, $\Sigma_{1,2}$, $\tau_{1,2}$ of non-Cartan part. The Weyl group of $\mathrm{SU}(3)$ is the permutation group Σ_3 and our basis is chosen so that the Maurer–Cartan equation exhibits Σ_3 symmetry. The isotropy representation of $\mathrm{SU}(3)/U(1)$ is decomposed as

$$\frac{\mathrm{su}(3)}{u(1)} = \mathbf{p}_1 \oplus \mathbf{p}_2 \oplus \mathbf{p}_3 \oplus \mathbf{p}_4, \tag{2.1}$$

where $\mathbf{p}_i$ ($i = 1$–4) are irreducible $U(1)$-modules with $\dim \mathbf{p}_1 = \dim \mathbf{p}_2 = \dim \mathbf{p}_3 = 2$ and $\dim \mathbf{p}_4 = 1$. Our metric ansatz is diagonal with respect to (2.1) for all t,

$$g = \mathrm{d}t^2 + a(t)^2(\sigma_1^2 + \sigma_2^2) + b(t)^2(\Sigma_1^2 + \Sigma_2^2) + c(t)^2(\tau_1^2 + \tau_2^2) + f(t)^2 T_A^2. \tag{2.2}$$

We have taken a quotient by the $U(1)$ subgroup generated by T_B. The vielbein of the above metric is

$$e^0 = \mathrm{d}t, \qquad e^1 = a(t)\sigma_1, \qquad e^2 = a(t)\sigma_2, \qquad e^3 = b(t)\Sigma_1, \qquad e^4 = b(t)\Sigma_2,$$
$$e^5 = c(t)\tau_1, \qquad e^6 = c(t)\tau_2, \qquad e^7 = f(t)T_A. \tag{2.3}$$

The spin connection ω_{ab} is obtained from the condition $De^a = \mathrm{d}e^a + \omega_{ab} \wedge e^b = 0$. We consider the octonionic self-duality of the spin connection

$$\omega_{ab} = \tfrac{1}{2}\Psi_{abcd}\omega_{cd}, \tag{2.4}$$

where totally anti-symmetric tensor Ψ_{abcd} is defined by the structure constants of octonions ψ_{abc} as follows:

$$\Psi_{abc0} = \psi_{abc}, \quad 1 \le a, b, c, \ldots, \le 7, \qquad \Psi_{abcd} = -\frac{1}{3!}\epsilon_{abcdefg}\psi_{efg}. \tag{2.5}$$

It can be shown that (2.4) implies the 4-form defined by

$$\Omega = \frac{1}{4!}\Psi_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d \tag{2.6}$$

is closed and the metric has Spin(7) holonomy [18]. Explicitly, the octonionic instanton equation in the present case is given by the structure constants:[1]

$$\psi_{abc} = +1 \quad \text{for} \quad (abc) = (721), (641), (135), (254), (263), (374), (765). \tag{2.7}$$

We obtain the following first-order differential equations:

$$\frac{\dot{a}}{a} = \frac{b^2 + c^2 - a^2}{abc} - \alpha_A\frac{f}{a^2}, \qquad \frac{\dot{b}}{b} = \frac{c^2 + a^2 - b^2}{abc} - \beta_A\frac{f}{b^2},$$
$$\frac{\dot{c}}{c} = \frac{a^2 + b^2 - c^2}{abc} - \gamma_A\frac{f}{c^2}, \qquad \frac{\dot{f}}{f} = \alpha_A\frac{f}{a^2} + \beta_A\frac{f}{b^2} + \gamma_A\frac{f}{c^2}, \tag{2.8}$$

where the parameters $\alpha_A, \beta_A, \gamma_A$ appearing in the Maurer–Cartan equation of T_A satisfy the "traceless" condition $\alpha_A + \beta_A + \gamma_A = 0$. These parameters have to be rational for the $U(1)$ subgroup generated by T_B to be a closed subgroup (see also Section 5). We assume this condition required by topological consistency in the following. Then there exists an integer N so that $(\alpha_A, \beta_A, \gamma_A) = (1/N)(n_1, n_2, n_3)$ and $\vec{n} := (n_1, n_2, n_3)$ are integers with no common divisor. Since N is eliminated by the rescaling $f \to Nf$, we may assume $(\alpha_A, \beta_A, \gamma_A) = (n_1, n_2, n_3)$ without any loss of generality. Our Spin(7) gravitational instanton equation is manifestly symmetric under the permutation group $\Sigma_3 = W(\mathrm{SU}(3))$, which can be regarded as the Weyl group of SU(3).

We can also derive the octonionic instanton equation (2.8) from the Lagrangian formulation. In the description of the extrinsic geometry of hypersurface, the shape operator $\mathcal{L}$ of the principal orbit $\mathrm{SU}(3)/U(1) \subset \tilde{M}$ satisfies the equation [17]

$$\dot{g}_t = 2g_t \circ \mathcal{L}. \tag{2.9}$$

[1] An appropriate permutation of the indices and the overall parity (sign) change are necessary to match our convention to the standard one. This parity change is an analog of the exchange of self-duality and anti-self-duality in four dimensions and related to the orientation of the manifold.

For the metric (2.2) it has a diagonal form

$$\mathcal{L}(t) = \mathrm{diag}\left(\frac{\dot{a}}{a}, \frac{\dot{a}}{a}, \frac{\dot{b}}{b}, \frac{\dot{b}}{b}, \frac{\dot{c}}{c}, \frac{\dot{c}}{c}, \frac{\dot{f}}{f}\right). \tag{2.10}$$

The Ricci-flatness condition then becomes [17]

$$\dot{\mathcal{L}} + (\mathrm{tr}\,\mathcal{L})\mathcal{L} - \mathrm{Ric} = 0, \tag{2.11}$$

$$\mathrm{tr}\,\dot{\mathcal{L}} + \mathrm{tr}(\mathcal{L}^2) = 0, \tag{2.12}$$

where Ric denotes the Ricci curvature of the metric g_t on $\mathrm{SU}(3)/U(1)$. Eq. (2.11) expresses the Ricci-flatness condition in directions tangent to the principal orbit, while (2.12) is obtained by considering the normal direction, i.e., t-direction. The Ricci-flatness of the mixed directions is automatically satisfied. This system of non-linear differential equations is described by the Lagrangian $L = T - V$:

$$T = ((\mathrm{tr}\,\mathcal{L})^2 - \mathrm{tr}(\mathcal{L}^2))\sqrt{\det g_t}, \qquad V = -R\sqrt{\det g_t}, \tag{2.13}$$

where $\det g_t = a^4b^4c^4f^2$ and R is the scalar curvature of g_t. After some calculation, we find

$$R = -2\left(\frac{a^2}{b^2c^2} + \frac{b^2}{a^2c^2} + \frac{c^2}{a^2b^2} - \frac{6}{a^2} - \frac{6}{b^2} - \frac{6}{c^2}\right) - 2f^2\left(\frac{n_1^2}{a^4} + \frac{n_2^2}{b^4} + \frac{n_3^2}{c^4}\right). \tag{2.14}$$

If we take the trace of (2.11) together with (2.12), we obtain

$$(\mathrm{tr}\,\mathcal{L})^2 - \mathrm{tr}(\mathcal{L}^2) - R = 0, \tag{2.15}$$

which gives a constraint $E = T + V = 0$ of this system. Therefore, the trajectories of "point particle" living on the level set $E = 0$ represent Ricci-flat Riemannian manifolds. Introducing a new time parameter τ defined by $\mathrm{d}t = a^2b^2c^2f\,\mathrm{d}\tau$, we can write the kinetic term as

$$T = \frac{1}{2}g_{ij}\frac{\mathrm{d}\alpha^i}{\mathrm{d}\tau}\frac{\mathrm{d}\alpha^j}{\mathrm{d}\tau}, \tag{2.16}$$

where the metric is given by

$$g_{ij} = \begin{pmatrix} 4 & 8 & 8 & 4 \\ 8 & 4 & 8 & 4 \\ 8 & 8 & 4 & 4 \\ 4 & 4 & 4 & 0 \end{pmatrix}, \tag{2.17}$$

and $\alpha^i = (\alpha, \beta, \gamma, \sigma)$ with $a = e^\alpha$, $b = e^\beta$, $c = e^\gamma$, $f = e^\sigma$. The potential V is expressed in terms of a superpotential W as

$$V = -\frac{1}{2}g^{ij}\frac{\partial W}{\partial \alpha^i}\frac{\partial W}{\partial \alpha^j} \tag{2.18}$$

with

$$W = 4abcf(a^2 + b^2 + c^2) + 2f^2(n_1b^2c^2 + n_2a^2c^2 + n_3a^2b^2). \tag{2.19}$$

Thus the Ricci-flatness condition follows from the gradient flow equation:

$$\frac{\mathrm{d}\alpha^i}{\mathrm{d}\tau} = g^{ij}\frac{\partial W}{\partial \alpha^j}, \tag{2.20}$$

which reproduces the instanton equation (2.8).

3. ALC solutions

Let us first make a change of variable defined by $\mathrm{d}r = f(t)\,\mathrm{d}t$ and take the following ansatz to solve the instanton equation (2.8):

$$\begin{aligned} a^2(r) &= \frac{2n_1}{\alpha_1 - \alpha_2}(r - \alpha_1)(r - \alpha_2), \qquad b^2(r) = \frac{2n_2}{\beta_1 - \beta_2}(r - \beta_1)(r - \beta_2), \\ c^2(r) &= \frac{2n_3}{\gamma_1 - \gamma_2}(r - \gamma_1)(r - \gamma_2), \end{aligned} \tag{3.1}$$

so that we have $a^2(r), b^2(r), c^2(r) \sim r^2$ as $r \to \infty$. The overall normalizations are fixed by the requirement that we can make a quadrature of the differential equation for $f(r)$ to obtain

$$f^2(r) = \frac{(r - \alpha_1)(r - \beta_1)(r - \gamma_1)}{(r - \alpha_2)(r - \beta_2)(r - \gamma_2)}. \tag{3.2}$$

Since $f(r) \sim$ constant as $r \to \infty$ in our ansatz, asymptotically there is an S^1 of a constant radius at infinity. Thus solutions obtained by this ansatz will give ALC metrics in the sense of Cvetič et al. [10]. We find that if the parameters obey

$$\alpha_1 - \alpha_2 = 2n_1, \qquad \beta_1 - \beta_2 = 2n_2, \qquad \gamma_1 - \gamma_2 = 2n_3, \tag{3.3}$$

and

$$\alpha_1 + \beta_1 = 2\gamma_2, \qquad \beta_1 + \gamma_1 = 2\alpha_2, \qquad \gamma_1 + \alpha_1 = 2\beta_2, \tag{3.4}$$

then the ansatz (3.1) gives a Spin(7) gravitational instanton. We have expressed the conditions in a completely Σ_3 symmetric manner. Note that due to the constraint $n_1+n_2+n_3 = 0$, one of the six conditions is redundant and we have one free parameter that corresponds to a translation of the radial coordinate r. After rescaling the radial coordinate $r \to r/\ell$ by an arbitrary positive parameter ℓ with dimensions of length, our ALC solutions can be written as

$$\begin{aligned} a^2(r) &= (r - \alpha_1\ell)(r - \alpha_2\ell), \qquad b^2(r) = (r - \beta_1\ell)(r - \beta_2\ell), \\ c^2(r) &= (r - \gamma_1\ell)(r - \gamma_2\ell), \qquad f^2(r) = \ell^2\frac{(r - \alpha_1\ell)(r - \beta_1\ell)(r - \gamma_1\ell)}{(r - \alpha_2\ell)(r - \beta_2\ell)(r - \gamma_2\ell)} \end{aligned} \tag{3.5}$$

with the conditions (3.3) and (3.4). The asymptotic form of the metric is

$$g \sim \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_6 + \ell^2 T_A^2, \tag{3.6}$$

where $\mathrm{d}\Omega_6$ is a homogeneous metric on the flag manifold $\mathrm{SU}(3)/U(1)\times U(1)$, which is the twistor space of $\mathbf{CP}(2)$.

Let us look at a few special examples, where a cancellation of a zero and a pole of the rational function f^2 takes place.

1. $\vec{n} = (1,-1,0)$

   In this case we can take $\gamma_1 = \gamma_2 = 0$ by a translation of r. Then the solution is

$$a^2(r) = (r-\tfrac{4}{3}\ell)(r+\tfrac{2}{3}\ell), \qquad b^2(r) = (r+\tfrac{4}{3}\ell)(r-\tfrac{2}{3}\ell), \qquad c^2(r) = r^2,$$
$$f^2(r) = \ell^2\frac{(r-\frac{4}{3}\ell)(r+\frac{4}{3}\ell)}{(r+\frac{2}{3}\ell)(r-\frac{2}{3}\ell)}. \tag{3.7}$$

   The solution has the same structure[2] as new G_2 metrics in [13] based on $S^3\times S^3$. The metric is regular in the region $r > 4\ell/3$. At the boundary $a^2\to 0$, $f^2\to 0$ but $b^2 = c^2 = 16\ell^2/9$. Since b^2 and c^2 approach the same boundary value, we have $\mathbf{CP}(2)$ as a singular orbit (see the next section).

2. $\vec{n} = (1,1,-2)$

   In this case $\alpha_i = \beta_i$ has to be satisfied and we obtain a reduced solution with $a^2 = b^2$. We take $\alpha_1 = \beta_1 = 1$. Then the solution is

$$a^2(r) = (r-\ell)(r+\ell), \qquad b^2(r) = a^2(r), \qquad c^2(r) = (r+3\ell)(r-\ell),$$
$$f^2(r) = \ell^2\frac{(r-\ell)(r+3\ell)}{(r+\ell)^2}. \tag{3.8}$$

   We see the solution has the same form as the simplest solution (denoted as $\mathbf{A}_8$) among new Spin(7) metrics in [10]. The metric is regular in the region $r > \ell$ and at $r = \ell$ all the coefficients are linearly vanishing. Thus the principal orbit $\mathrm{SU}(3)/U(1)$ collapses to a point and the manifold has curvature singularities at $r = \ell$, since $\mathrm{SU}(3)/U(1)$ is not homeomorphic to S^7.

3. $\vec{n} = (2,-1,-1)$

   This is obtained from the second example simply by the sign flip and a permutation, but the global topology is different as we will see shortly. In the same way as above the solution is

$$a^2(r) = (r-3\ell)(r+\ell), \qquad b^2(r) = (r-\ell)(r+\ell), \qquad c^2(r) = b^2(r),$$
$$f^2(r) = \ell^2\frac{(r+\ell)(r-3\ell)}{(r-\ell)^2}. \tag{3.9}$$

   The regular region is $r > 3\ell$ and in contrast to the second example we have $\mathbf{CP}(2)$ with

[2] It is only at the level of solutions to the first-order system and does not mean the geometry is the same, since the starting coset space is different.

finite volume at the boundary. This solution corresponds to the solution denoted $\mathbf{B}_8$ in [10].

4. Perturbation around the singular orbit

In this section, we will give a perturbative expansion in a small neighborhood of the singular orbit ($t = 0$) for the Spin(7) gravitational instanton equation (2.8). A mathematical foundation may be found in [17]. Our metric is written in the form $g = \mathrm{d}t^2 + g_t$, where g_t ($t \geq 0$) is a one-parameter family of SU(3)-invariant metrics on the principal orbit $\mathrm{SU}(3)/U(1)$. We assume that near the boundary $t = 0$ the orbit is locally of the form

$$\frac{\mathrm{SU}(3)}{U(1)} \to S^3 \times \frac{\mathrm{SU}(3)}{U(2)}, \tag{4.1}$$

where S^3 denotes a round 3-sphere whose radius tends to zero at $t = 0$, and the singular orbit $\mathrm{SU}(3)/U(2)$ is a complex projective space $\mathbf{CP}(2)$ whose size remains non-vanishing at $t = 0$. Thus, if we choose the rate of collapse of the S^3 factor appropriately, the manifold approaches $\mathbf{R}^4 \times \mathbf{CP}(2)$ at short distance; it has topologically the same local behavior as the hyper-Kähler manifold $T^*\mathbf{CP}(2)$. In general a singular orbit gives a singularity of the instanton equation and we have no smooth solution at the singularity. However, in our case, the very geometric nature of the equation allows a smooth solution in a neighborhood around a singular orbit.

Due to the Σ_3 symmetry of our instanton equation, we have three types of possible boundary conditions for the limit $t \to 0$:

$$g \to \mathrm{d}t^2 + t^2\left(\frac{T_A^2}{n_1^2} + \sigma_1^2 + \sigma_2^2\right) + m^2(\Sigma_1^2 + \Sigma_2^2 + \tau_1^2 + \tau_2^2), \tag{4.2}$$

$$g \to \mathrm{d}t^2 + t^2\left(\frac{T_A^2}{n_2^2} + \Sigma_1^2 + \Sigma_2^2\right) + m^2(\sigma_1^2 + \sigma_2^2 + \tau_1^2 + \tau_2^2), \tag{4.3}$$

$$g \to \mathrm{d}t^2 + t^2\left(\frac{T_A^2}{n_3^2} + \tau_1^2 + \tau_2^2\right) + m^2(\sigma_1^2 + \sigma_2^2 + \Sigma_1^2 + \Sigma_2^2), \tag{4.4}$$

where m is a scale parameter corresponding to the size of $\mathbf{CP}(2)$. By choosing a boundary condition the Σ_3 symmetry is broken to the $\mathbf{Z}_2$ symmetry. For each choice of the boundary condition which specifies a $\mathbf{CP}(2)$ embedded in $\mathrm{SU}(3)/U(1)$, the unit volume element $\vec{v}_\alpha$ ($\alpha = 1, 2, 3$) of the $\mathbf{CP}(2)$ is given by

$$\vec{v}_1 = e^3 \wedge e^4 \wedge e^5 \wedge e^6, \qquad \vec{v}_2 = e^1 \wedge e^2 \wedge e^5 \wedge e^6, \qquad \vec{v}_3 = e^1 \wedge e^2 \wedge e^3 \wedge e^4, \tag{4.5}$$

respectively. For all three cases the calibration Ω is given by (2.6) satisfies the equation

$|\Omega(\vec{v}_\alpha)| = 1$. We therefore see that the singular orbit $\mathbf{CP}(2)$ appearing at the boundary is a Cayley submanifold (supersymmetric four-cycles) in Spin(7) holonomy manifold.

The solutions with these three boundary conditions and the corresponding $\mathbf{CP}(2)$'s are permuted by the action of Σ_3. For concreteness, we will consider from now on the first boundary condition (4.2). The quantity $\Sigma_1^2 + \Sigma_2^2 + \tau_1^2 + \tau_2^2$ represents the Fubini–Study metric on $\mathbf{CP}(2)$ and $T_A^2/n_1^2 + \sigma_1^2 + \sigma_2^2$ locally the metric on the unit 3-sphere. More precisely, the metric g would have an orbifold singularity at $t = 0$ unless we choose the value of $\vec{n}$ appropriately, since the latter represents the metric on the Lens space $S^3/\mathbf{Z}_n$ in general rather than S^3 globally (see the next section). The perturbative series expansion around the singular orbit $\mathbf{CP}(2)$ yields

$$
\begin{aligned}
a(t) &= t\left(1 - \frac{1}{2}(Q+1)\left(\frac{t}{m}\right)^2 + \cdots\right),\\
b(t) &= m\left(1 + \frac{1}{6}\left(4 - \frac{n_2}{n_1}\right)\left(\frac{t}{m}\right)^2 + \cdots\right),\\
c(t) &= m\left(1 + \frac{1}{6}\left(4 - \frac{n_3}{n_1}\right)\left(\frac{t}{m}\right)^2 + \cdots\right),\\
f(t) &= \frac{t}{n_1}\left(1 + Q\left(\frac{t}{m}\right)^2 + \cdots\right).
\end{aligned}
\tag{4.6}
$$

It should be noticed that the solution is not uniquely determined by the boundary condition and it includes an additional free parameter Q. The expansion (4.6) is a consequence of the assumption (4.2) and valid for any solution that is smooth around the singular orbit. However, it is not at all clear that the local perturbative solution (4.6) can be extended to a global complete metric. There should be some bound on Q that may depend on a choice of $\vec{n}$. One can obtain an example of local solution that extends to a complete metric by setting $\vec{n} = (1, 1, -2)$[3] and $Q = -2/3$. Then it can be extended to the Calabi hyper-Kähler metric on $T^*\mathbf{CP}(2)$ of Sp(2) holonomy [9,15]

$$
\begin{aligned}
&a^2(r) = \frac{1}{2}(r^2 - m^2), \qquad b^2(r) = \frac{1}{2}(r^2 + m^2), \qquad c^2(r) = r^2,\\
&f^2(r) = \frac{r^2}{4}\left(1 - \left(\frac{m}{r}\right)^4\right)
\end{aligned}
\tag{4.7}
$$

with $\mathrm{d}t = \mathrm{d}r/(1 - (m/r)^4)^{1/2}$. Note that the asymptotic behavior of the Calabi metric at infinity is different from the ALC metrics.

We can also find the specialization which reproduces the ALC metrics described in the previous section. Let us consider the solution (3.5) whose radial coordinate is constrained

[3] With this choice of $\vec{n}$ the ALC solution cannot have a singular orbit with finite volume (see the second example in Section 3).

to be $r \geq \alpha_1 \ell$. From the conditions (3.3) and (3.4) we have

$$\alpha_1 - \alpha_2 = 2n_1, \qquad \alpha_1 - \beta_1 = \tfrac{4}{3}(n_1 - n_2), \qquad \alpha_1 - \beta_2 = \tfrac{2}{3}(n_1 - n_3),$$
$$\alpha_1 - \gamma_1 = \tfrac{4}{3}(n_1 - n_3), \qquad \alpha_1 - \gamma_2 = \tfrac{2}{3}(n_1 - n_2). \tag{4.8}$$

Thus if we choose the parameter $\vec{n}$ as

$$n_1 > 0, \qquad n_1 > n_2, \qquad n_1 > n_3, \tag{4.9}$$

then the solution $\{a, b, c, f\}$ is non-vanishing in the region $r > \alpha_1 \ell$, and so the ALC metric is non-singular. The behavior of the metric near $r = \alpha_1 \ell$ is given by

$$g \to \mathrm{d}\rho^2 + \rho^2 \left(\frac{T_A^2}{n_1^2} + \sigma_1^2 + \sigma_2^2 \right) + \ell^2 (\alpha_1 - \beta_1)(\alpha_1 - \beta_2)(\Sigma_1^2 + \Sigma_2^2 + \tau_1^2 + \tau_2^2), \tag{4.10}$$

where $\rho^2 = \ell(\alpha_1 - \alpha_2)(r - \alpha_1 \ell)$. Setting $m = \ell\sqrt{(\alpha_1 - \beta_1)(\alpha_1 - \beta_2)}$, we reproduce Eq. (4.2) and higher order calculations yield the relation

$$Q = -\frac{2}{27}\left(13 + \frac{2n_2 n_3}{n_1^2} \right). \tag{4.11}$$

The parameter in the perturbative solution Q implies a possibility of non-trivial deformations of the ALC solutions (3.5). Although we have not been able to find general solutions in closed form, numerical simulations of the instanton equation indicate a family of global solutions under some conditions of Q. For example, in the case of $\vec{n} = (1, -1, 0)$, the condition is given by $Q \leq -0.35$ approximately and the exact solution (3.7) with $Q = -26/27$ is included in this region. The existence of more general solutions is also supported by a similar analysis for the coset space $\mathrm{Sp}(2)/\mathrm{Sp}(1)$, where the general solutions of complete metric have been obtained [10]. In Appendix B, it is shown briefly how we can accommodate a local perturbative analysis around the singular orbit of $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ model to the global solutions in [10].

5. The issue of global topology

Let us consider the global topology of solutions with boundary condition (4.2) by calculating explicitly the 1-forms σ_1, σ_2 and T_A describing the metric on the unit 3-sphere S^3 locally. Our calculation shows that near the boundary the topology of the principal orbit is in general $S^3/\mathbf{Z}_{n_1} \times \mathbf{CP}(2)$ rather than $S^3 \times \mathbf{CP}(2)$. The integer n_1 comes in here since we take the first boundary condition (4.2). For other boundary conditions (4.3) and (4.4), n_1 is replaced by n_2 or n_3 accordingly.

To consider the topology of the fiber over a point of the base space $\mathbf{CP}(2)$ we fix the coordinates on $\mathbf{CP}(2)$. Then $\Sigma_i = \tau_j = 0$ on the fiber and the SU(3) Maurer–Cartanpar

equation reduces to the following form:

$$\mathrm{d}\sigma_1 = \kappa_A T_A \wedge \sigma_2 + \kappa_B T_B \wedge \sigma_2, \qquad \mathrm{d}\sigma_2 = -\kappa_A T_A \wedge \sigma_1 - \kappa_B T_B \wedge \sigma_1,$$
$$\mathrm{d}T_A = 2\alpha_A \sigma_1 \wedge \sigma_2, \qquad \mathrm{d}T_B = 2\alpha_B \sigma_1 \wedge \sigma_2. \tag{5.1}$$

In fact, this Maurer–Cartan equation comes from the following choice of the generators E_α ($\alpha = 1, 2$ or A, B) of $U(2)$:

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix},$$
$$E_A = -\frac{1}{\Delta}\begin{pmatrix} \alpha_B & 0 & 0 \\ 0 & \beta_B & 0 \\ 0 & 0 & \gamma_B \end{pmatrix}, \qquad E_B = \frac{1}{\Delta}\begin{pmatrix} \alpha_A & 0 & 0 \\ 0 & \beta_A & 0 \\ 0 & 0 & \gamma_A \end{pmatrix}. \tag{5.2}$$

By expanding a left invariant 1-form ω of the subgroup $U(2) \subset \mathrm{SU}(3)$

$$\omega = \mathrm{i}(\sigma_1 E_1 + \sigma_2 E_2 + T_A E_A + T_B E_B), \tag{5.3}$$

and using the relations in Appendix A, we can check the equation $\mathrm{d}\omega + \omega \wedge \omega = 0$ implies (5.1).

The left invariant 1-form ω is represented by $\omega = g^{-1}\,\mathrm{d}g$ in terms of a group element $g \in U(2)$. To parameterize the group element g we use Euler angles (θ, ϕ, ψ) with the range

$$0 \le \theta < \pi, \qquad 0 \le \phi < 2\pi, \qquad 0 \le \psi < 4\pi \tag{5.4}$$

of $\mathrm{SU}(2)$ and a $U(1)$ coordinate χ defined by

$$U_B(1) = \exp(\tfrac{1}{2}\mathrm{i}\Delta\chi E_B) = \mathrm{diag}(\mathrm{e}^{\mathrm{i}n_1\chi/2}, \mathrm{e}^{\mathrm{i}n_2\chi/2}, \mathrm{e}^{\mathrm{i}n_3\chi/2}), \tag{5.5}$$

where $(n_1, n_2, n_3) := (\alpha_A, \beta_A, \gamma_A)$ are integers with no common divisor[4] satisfying the traceless condition $n_1 + n_2 + n_3 = 0$. With these coordinates the group element g is given by

$$g(\phi, \theta, \psi, \chi) = \mathrm{e}^{\mathrm{i}(\phi/2)E_3}\,\mathrm{e}^{\mathrm{i}(\theta/2)E_2}\,\mathrm{e}^{\mathrm{i}(\psi/2)E_3}\,\mathrm{e}^{\mathrm{i}(\Delta\chi/2)E_B}$$
$$= \begin{pmatrix} \mathrm{e}^{\mathrm{i}n_1\chi/2} & 0 & 0 \\ 0 & \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}(\psi+\phi+n_2\chi)/2} & \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}(\psi-\phi-n_3\chi)/2} \\ 0 & -\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}(\psi-\phi+n_2\chi)/2} & \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}(\psi+\phi-n_3\chi)/2} \end{pmatrix}, \tag{5.6}$$

where

$$E_3 = \alpha_A E_A + \alpha_B E_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{5.7}$$

[4] See the discussion below Eq. (2.8).

We now obtain the 1-forms from (5.3)

$$\begin{aligned}\sigma_1 &= \tfrac{1}{2}\sin\theta\cos\left(\psi+\tfrac{1}{2}(n_2-n_3)\chi\right)\mathrm{d}\phi-\tfrac{1}{2}\sin\left(\psi+\tfrac{1}{2}(n_2-n_3)\chi\right)\mathrm{d}\theta,\\ \sigma_2 &= \tfrac{1}{2}\sin\theta\sin\left(\psi+\tfrac{1}{2}(n_2-n_3)\chi\right)\mathrm{d}\phi+\tfrac{1}{2}\cos\left(\psi+\tfrac{1}{2}(n_2-n_3)\chi\right)\mathrm{d}\theta,\\ T_A &= \tfrac{1}{2}n_1(\mathrm{d}\psi+\cos\theta\,\mathrm{d}\phi).\end{aligned} \tag{5.8}$$

These 1-forms give a metric

$$\frac{T_A^2}{n_1^2}+\sigma_1^2+\sigma_2^2=\frac{1}{4}(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)+\frac{1}{4}(\mathrm{d}\psi+\cos\theta\,\mathrm{d}\phi)^2 \tag{5.9}$$

on the coset space $U(2)/U_B(1)$ whose topology is locally S^3, but not globally S^3. In fact for each integer $p=0,1,\ldots,n_1-1$ we can find an integer q such that we have

$$g\left(\phi,\theta,\psi+\frac{4\pi p}{n_1},\chi+\frac{4\pi q}{n_1}\right)=g(\phi,\theta,\psi,\chi). \tag{5.10}$$

Thus the angle ψ is identified with $\psi+4\pi p/n_1$, which implies Eq. (5.9) expresses the metric on $S^3/\mathbf{Z}_{n_1}$. This is the total space of $U(1)$-bundle over S^2 with a connection (Wu–Yang monopole potential) T_A and the topological index is given by Quiros and Mittelbrunn [19]

$$\frac{1}{2\pi}\int_{S^2}\mathrm{d}T_A=n_1. \tag{5.11}$$

In conclusion, if we normalize $\vec{n}=(n_1,n_2,n_3)$ as integers with no common divisor, then global solutions to the instanton equation (2.8) with the boundary condition (4.2) describe manifolds of special holonomy which behave like $\mathbf{R}^4/\mathbf{Z}_{n_1}\times\mathbf{CP}(2)$ near the singular orbit. For example the Calabi metric which has $\vec{n}=(1,1,-2)$ is a hyper-Kähler metric on $T^*\mathbf{CP}(2)$ with a fiber $\mathbf{C}^2=\mathbf{R}^4$. Concerning the examples of ALC solutions in Section 3, the solution (3.7) with $\vec{n}=(1,-1,0)$ has a trivial fiber $\mathbf{R}^4$, while the fiber of the solution (3.9) with $\vec{n}=(2,-1,-1)$ is $\mathbf{R}^4/\mathbf{Z}_2$. By the constraint (4.9) on $\vec{n}$, among our ALC solutions (3.5) only the solution (3.7) gives a complete metric of Spin(7) holonomy without orbifold singularities.

6. L^2-normalizable harmonic 4-forms

In this section, we consider L^2-normalizable harmonic 4-forms on ALC manifolds of Spin(7) holonomy. For the metric (2.2) of cohomogeneity one, we assume the following self-dual 4-form G:

$$\begin{aligned}G=\ &u_1(t)(e^{0567}+e^{1243})+u_2(t)(e^{0473}+e^{2561})+u_3(t)(e^{0127}+e^{3654})\\ &+u_4(t)(e^{0315}+e^{4267}+e^{0524}+e^{6371}+e^{0461}+e^{7325}+e^{0362}+e^{5471}),\end{aligned} \tag{6.1}$$

where $e^{abcd}=e^a\wedge e^b\wedge e^c\wedge e^d$. If $u_A=1$ for all A, then the 4-form G is equal to the

calibration Ω. The closeness condition $\mathrm{d}G = 0$ is expressed by the equations

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(a^2b^2u_1) = 4abcu_4 - 2n_1fb^2u_2 - 2n_2fa^2u_3,\\
&\frac{\mathrm{d}}{\mathrm{d}t}(a^2c^2u_2) = 4abcu_4 - 2n_1fc^2u_1 - 2n_3fa^2u_3,\\
&\frac{\mathrm{d}}{\mathrm{d}t}(b^2c^2u_3) = 4abcu_4 - 2n_2fc^2u_1 - 2n_3fb^2u_2,\\
&\frac{\mathrm{d}}{\mathrm{d}t}(abcfu_4) = f(a^2u_3 + b^2u_2 + c^2u_1).
\end{aligned} \tag{6.2}$$

The instanton equation leads to a linear relation as the first integral

$$u_1 + u_2 + u_3 + 4u_4 = k, \tag{6.3}$$

where the constant k should be chosen to zero by the L^2-normalizability of the 4-form G. This choice is also consistent with the criterions for unbroken supersymmetry[5] for compactifications of M theory on manifolds of Spin(7) holonomy [20]:

$$*G = G, \qquad \omega = 0, \qquad G \wedge \Omega = 0, \tag{6.4}$$

where ω is the 2-form defined by $\omega = G_{abcd}\Psi_{abcf}e^d \wedge e^f$. In fact, our ansatz satisfies $*G = G, \omega = 0$ automatically and the last equation requires precisely the relation (6.3) with $k = 0$. Thus following [8–10], one can construct supersymmetric M2-branes using the L^2-normalizable solutions of (6.2).

In order to obtain explicit solutions of (6.2), we take the ALC solution (3.7) as the background metric. Then the radial coordinate r runs from the singular orbit at $r = (4/3)\ell$ to infinity. It is convenient to introduce the new variables

$$u = b^2c^2u_3 - a^2c^2u_2, \qquad v = b^2c^2u_3 + a^2c^2u_2, \qquad w = a^2b^2u_1. \tag{6.5}$$

After eliminating u_4 by (6.3) and further taking derivatives of the first-order equations (6.2), we obtain the Fuchs type differential equation:[6]

$$\frac{\mathrm{d}^3}{\mathrm{d}r^3}u + p_1(r)\frac{\mathrm{d}^2}{\mathrm{d}r^2}u + p_2(r)\frac{\mathrm{d}}{\mathrm{d}r}u + p_3(r)u = 0, \tag{6.6}$$

where

$$\begin{aligned}
p_1(r) &= \frac{8(2560 + 12384r^2 - 29160r^4 + 10206r^6 + 6561r^8)}{81r(r+2/3)(r-2/3)(r+4/3)(r-4/3)(9r^2+20)(9r^2-8)},\\
p_2(r) &= \frac{2(-10240 + 26496r^2 - 116640r^4 + 72900r^6 + 32805r^8)}{81r^2(r+2/3)(r-2/3)(r+4/3)(r-4/3)(9r^2+20)(9r^2-8)},\\
p_3(r) &= -\frac{48r(9r^2+104)}{(r+2/3)(r-2/3)(r+4/3)(r-4/3)(9r^2+20)(9r^2-8)}.
\end{aligned} \tag{6.7}$$

[5] We thank Katrin Becker for pointing out an error in the original version.

[6] We set the length parameter ℓ in the solution (3.7) to unity for convenience.

This equation can be integrated by imposing the regularity of the solution at the regular singular point $r = 4/3$ and we find a solution

$$u(r) = \frac{27r^2(9r^2 - 40)}{(r + 2/3)(r - 2/3)(r + 4/3)^2}. \tag{6.8}$$

The remaining functions v and w are given by taking derivatives of (6.8). Finally, we obtain an L^2-normalizable harmonic 4-form G in the region $r \geq 4/3$:

$$\begin{aligned}
u_1 &= \frac{2(160 - 72r^2 + 243r^3 + 81r^4)}{r(r + 2/3)^2(r - 2/3)^2(r + 4/3)^3}, \\
u_2 &= -\frac{3(32 + 48r - 72r^2 + 135r^3)}{2r^3(r - 2/3)(r + 2/3)^2(r + 4/3)}, \\
u_3 &= \frac{512 + 768r - 1440r^2 - 4752r^3 + 648r^4 + 243r^5}{6r^3(r + 2/3)(r - 2/3)^2(r + 4/3)^3}, \\
u_4 &= \frac{64 + 144r + 180r^2 - 81r^3}{r^3(r + 2/3)(r - 2/3)(r + 4/3)^3}.
\end{aligned} \tag{6.9}$$

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Appendix A

The spin connection on the coset space $\mathrm{SU}(3)/U(1)$ is obtained by the Maurer–Cartan equation for the left invariant 1-forms of SU(3). To derive the instanton equation in Σ_3 symmetric form, we here present the Σ_3 symmetric SU(3) Maurer–Cartan equation:

$$\begin{aligned}
\mathrm{d}\sigma_1 &= \Sigma_1 \wedge \tau_1 - \Sigma_2 \wedge \tau_2 + \kappa_A T_A \wedge \sigma_2 + \kappa_B T_B \wedge \sigma_2, \\
\mathrm{d}\sigma_2 &= -\Sigma_1 \wedge \tau_2 - \Sigma_2 \wedge \tau_1 - \kappa_A T_A \wedge \sigma_1 - \kappa_B T_B \wedge \sigma_1, \\
\mathrm{d}\Sigma_1 &= \tau_1 \wedge \sigma_1 - \tau_2 \wedge \sigma_2 + \mu_A T_A \wedge \Sigma_2 + \mu_B T_B \wedge \Sigma_2, \\
\mathrm{d}\Sigma_2 &= -\tau_1 \wedge \sigma_2 - \tau_2 \wedge \sigma_1 - \mu_A T_A \wedge \Sigma_1 - \mu_B T_B \wedge \Sigma_1, \\
\mathrm{d}\tau_1 &= \sigma_1 \wedge \Sigma_1 - \sigma_2 \wedge \Sigma_2 + \nu_A T_A \wedge \tau_2 + \nu_B T_B \wedge \tau_2, \\
\mathrm{d}\tau_2 &= -\sigma_1 \wedge \Sigma_2 - \sigma_2 \wedge \Sigma_1 - \nu_A T_A \wedge \tau_1 - \nu_B T_B \wedge \tau_1, \\
\mathrm{d}T_A &= 2\alpha_A \sigma_1 \wedge \sigma_2 + 2\beta_A \Sigma_1 \wedge \Sigma_2 + 2\gamma_A \tau_1 \wedge \tau_2, \\
\mathrm{d}T_B &= 2\alpha_B \sigma_1 \wedge \sigma_2 + 2\beta_B \Sigma_1 \wedge \Sigma_2 + 2\gamma_B \tau_1 \wedge \tau_2.
\end{aligned} \tag{A.1}$$

This form of the Maurer–Cartan equation is symmetric under the (cyclic) permutation of $(\sigma_i, \Sigma_i, \tau_i)$.[7] We have introduced parameters $\alpha, \beta, \gamma, \kappa, \mu, \nu$, which describe the "coupling" of the Cartan generators $\{T_A, T_B\}$. The condition $d^2 = 0$, or the Jacobi identity implies the following constraints for them:

$$\begin{aligned}
&\alpha_A + \beta_A + \gamma_A = \alpha_B + \beta_B + \gamma_B = 0, \qquad \kappa_A + \mu_A + \nu_A = \kappa_B + \mu_B + \nu_B = 0,\\
&\kappa_A\beta_A + \kappa_B\beta_B = \kappa_A\gamma_A + \kappa_B\gamma_B = 1, \qquad \mu_A\gamma_A + \mu_B\gamma_B = \mu_A\alpha_A + \mu_B\alpha_B = 1,\\
&\nu_A\alpha_A + \nu_B\alpha_B = \nu_A\beta_A + \nu_B\beta_B = 1.
\end{aligned} \tag{A.2}$$

These equations can be solved in the form

$$\begin{aligned}
&\kappa_A = \frac{1}{\Delta}(\beta_B - \gamma_B), \qquad \kappa_B = -\frac{1}{\Delta}(\beta_A - \gamma_A), \qquad \mu_A = -\frac{1}{\Delta}(\alpha_B - \gamma_B),\\
&\mu_B = \frac{1}{\Delta}(\alpha_A - \gamma_A), \qquad \nu_A = \frac{1}{\Delta}(\alpha_B - \beta_B), \qquad \nu_B = -\frac{1}{\Delta}(\alpha_A - \beta_A)
\end{aligned} \tag{A.3}$$

with $\Delta = \beta_A\alpha_B - \alpha_A\beta_B$ leaving four free parameters $(\alpha_{A,B}, \beta_{A,B})$. We may further put the "orthogonality" conditions

$$\alpha_A\alpha_B + \beta_A\beta_B + \gamma_A\gamma_B = 0, \qquad \kappa_A\kappa_B + \mu_A\mu_B + \nu_A\nu_B = 0, \tag{A.4}$$

which reduces one parameter. A standard choice of parameters (cf. [9]) is

$$\begin{aligned}
&(\alpha_A, \beta_A, \gamma_A) = (1, 1, -2), \qquad (\alpha_B, \beta_B, \gamma_B) = (1, -1, 0),\\
&{}_A, \mu_A, \nu_A) = (-1/2, -1/2, 1), \qquad (\kappa_B, \mu_B, \nu_B) = (-3/2, 3/2, 0),
\end{aligned} \tag{A.5}$$

and the remaining three parameters correspond to the two scalings of each $T_{A,B}$ and the overall rotation.

Appendix B

In this appendix, we present a perturbative analysis of the Spin(7) gravitational instanton equation based on the coset space Sp(2)/Sp(1) and discuss the relation to the exact regular solutions constructed in [6,7,10]. Let us write the Spin(7) metric in the form

$$g = dt^2 + a(t)^2(\Sigma_1^2 + \Sigma_2^2) + b(t)^2\sigma^2 + c(t)^2(\tau_1^2 + \tau_2^2 + \tau_3^2 + \tau_4^2). \tag{B.1}$$

Here $\{\Sigma_a, \sigma, \tau_i\}$ are 1-forms defined by using the Maurer–Cartan forms of Sp(2) [10] and $\{a, b, c\}$ are functions of the radial variable t associated with the decomposition of the isotropy representation $\mathrm{sp}(2)/\mathrm{sp}(1) = \mathbf{p}_1 \oplus \mathbf{p}_2 \oplus \mathbf{p}_3$ with dimensions 2, 1 and 4, respectively. The instanton equation is given by

$$\dot{a} = 1 - \frac{b}{2a} - \frac{a^2}{c^2}, \qquad \dot{b} = \frac{b^2}{2a^2} - \frac{b^2}{c^2}, \qquad \dot{c} = \frac{a}{c} + \frac{b}{2c}. \tag{B.2}$$

[7] To compare with the one in [9], we make an exchange $\Sigma_1 \leftrightarrow \Sigma_2$ and a change of notation and sign $(\nu_1, \nu_2) \to (\tau_1, -\tau_2)$.

We now assume that near $t = 0$ the principal orbit is locally of the form

$$\frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)} \to S^3 \times S^4, \tag{B.3}$$

where S^3 collapses as $t \to 0$, while S^4 remains with a finite radius m at $t = 0$. Then on the singular orbit the boundary condition for the metric functions can be written as

$$a(t) \to \tfrac{1}{2}t, \qquad b(t) \to \tfrac{1}{2}t, \qquad c(t) \to m \tag{B.4}$$

for $t \to 0$. The perturbative solution of (B.2) with the boundary condition (B.4) is

$$a(t) = \frac{t}{2}\left(1 + \frac{1}{2}\tilde{Q}\left(\frac{t}{m}\right)^2 + \cdots\right), \quad b(t) = \frac{t}{2}\left(1 - \frac{1}{2}(2\tilde{Q}+1)\left(\frac{t}{m}\right)^2 + \cdots\right),$$
$$c(t) = m\left(1 + \frac{3}{8}\left(\frac{t}{m}\right)^2 + \cdots\right). \tag{B.5}$$

This solution includes a free parameter $\tilde{Q}$ in addition to the scale parameter m. Thus the structure is very similar to that of the solution (4.6), although the geometrical setting is different. Fortunately, in this case, all regular Spin(7) metrics are known in the closed form [6,7,10] and so we can read the condition for $\tilde{Q}$ admitting global solutions. By making the power series expansion of the known solutions at $t = 0$ we find that the global solutions can arise in the parameter region $\tilde{Q} \geq -1/3$: the perturbative solutions lift to the metrics of Spin(7) holonomy defined on the bundle of chiral spinors over S^4. More precisely, the solutions in the regions: (a) $-1/3 < \tilde{Q} < 0$, (b) $\tilde{Q} = 0$, (c) $\tilde{Q} > 0$ lift to the ALC metrics $\mathbf{B}_8^+$, $\mathbf{B}_8$ and $\mathbf{B}_8^-$, respectively, using the notation of Cvetič et al. [10]. The boundary $\tilde{Q} = -1/3$ corresponds to the metric of Spin(7) holonomy obtained in [6,7], and the limit $\tilde{Q} \to \infty$ reduces to the metric of G_2 holonomy on the $\mathbf{R}^3$ bundle over S^4 [6,7]. The ALC solutions in this paper corresponds to the solution $\mathbf{B}_8$ and we expect there are deformations like $\mathbf{B}_8^+$, and $\mathbf{B}_8^-$ in our case too.

References

[1] B. Acharya, On realizing $N = 1$ super Yang–Mills in M theory. hep-th/0011089.
[2] M. Atiyah, J. Maldacene, C. Vafa, An M-theory flop as a large N duality. hep-th/0011256.
[3] J. Gomis, D-brane, holonomy and M-theory. hep-th/0103115.
[4] J.D. Edelstein, C. Núñez, D6 branes and M-theory geometrical transitions from gauged supergravity. hep-th/0103167.
[5] M. Atiyah, E. Witten, M-theory dynamics on a manifold of G_2 holonomy. hep-th/0107177.
[6] R. Bryant, S. Salamon, On the construction of some complete metrics with exceptional holonomy, Duke Math. J. 58 (1989) 829.
[7] G.W. Gibbons, D.N. Page, C.N. Pope, Einstein metrics on S^3, R^3 and R^4 bundles, Commun. Math. Phys. 127 (1990) 529.
[8] M. Cvetič, G.W. Gibbons, H. Lü, C.N. Pope, Ricci-flat metrics: harmonic forms and brane resolutions. hep-th/0012011.
[9] M. Cvetič, G.W. Gibbons, H. Lü, C.N. Pope, Hyper-Kähler Calabi metrics, L^2 harmonic forms, resolved M2-branes and $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence. hep-th/0102185.

[10] M. Cvetič, G.W. Gibbons, H. Lü, C.N. Pope, New complete non-compact Spin(7) manifolds. hep-th/0103155; M. Cvetič, G.W. Gibbons, H. Lü, C.N. Pope, New cohomogeneity one metrics with Spin(7) holonomy. math.DG/0105119.
[11] M. Cvetič, G.W. Gibbons, H. Lü, C.N. Pope, Supersymmetric M3-branes and G_2 manifolds. hep-th/0106026.
[12] M. Cvetič, G.W. Gibbons, J.T. Liu, H. Lü, C.N. Pope, A new fractional D2-brane, G_2 holonomy and T-duality. hep-th/0106162.
[13] A. Brandhuber, J. Gomis, S.S. Gubser, S. Gukov, Gauge theory at large N and new G_2 holonomy metrics. hep-th/0106034.
[14] Y. Konishi, M. Naka, Coset construction of Spin(7), G_2 gravitational instanton. hep-th/0104208.
[15] A. Dancer, A. Swann, Hyper-Kähler metrics of cohomogeneity one, J. Geom. Phys. 21 (1997) 218.
[16] A. Dancer, M.Y. Wang, Kähler–Einstein metrics of cohomogeneity one, Math. Ann. 312 (1998) 549.
[17] J.-H. Eschenburg, M.Y. Wang, The initial value problem for cohomogeneity one Einstein metrics, J. Geom. Anal. 10 (2000) 109.
[18] I. Bakas, E.G. Floratos, A. Kehagias, Octonionic gravitational instanton, Phys. Lett. B 445 (1998) 69. hep-th/9810042.
[19] M. Quiros, J.R. Mittelbrunn, On the topological meaning of magnetic charge quantization, J. Math. Phys. 23 (1982) 873.
[20] K. Becker, A note on compactifications on Spin(7)-holonomy manifolds, JHEP 0011 (2000) 029. hep-th/0011114.
[21] M. Cvetič, G.W. Gibbons, H. Lü, C.N. Pope, Cohomogeneity one manifolds of Spin(7) and G_2 holonomy. hep-th/0108245.
[22] S. Gukov, J. Sparks, M-Theory on Spin(7) manifolds, I. hep-th/0109025.